Gauge-invariant reference section and geometric phase

1995 J. Phys. A: Math. Gen. 28 2087

(http://iopscience.iop.org/0305-4470/28/7/027)

J. Phys. A: Math. Gen. **28** (1995) 2087–2094. Printed in the UK

Gauge-invariant reference section and geometric phase

Arun Kumar Pati

Theoretical Physics Division, 5th floor, Central Complex, Bhabha Atomic Research Centre, Bombay-400 085, India

Received 3 March 1994, in final form 7 November 1994

Abstract. We use a gauge-invariant 'reference section' and define the geometric phase for all quantum evolutions in a closed form. This geometric phase is obtained by integrating the inner product of the 'reference section' and its path derivative along the evolution curve which is valid for non-cyclic, non-unitary and non-Schrödinger evolutions of quantum systems. Two non-trivial examples are studied to realize our new expression.

Recent years have seen the successful prediction and generalization of the Berry phase [1, 2] from adiabatic, parametric variation to cyclic quantum evolutions [3, 4] and, next, non-unitary and non-cyclic evolutions [5], which has left its imprint on almost all areas of physics [6]. Simon [2] interpreted the adiabatic Berry phase as the holonomy transformation in parallel transporting the adiabatic eigenstate in parameter space. The present author has interpreted the Aharonov and Anandan phase in terms of the integral of the contracted length of the curve [7, 9]. Until today the geometric phase for non-cyclic evolution has been defined by closing the end-points by the shortest geodesic. It is an indirect definition, because if the end-points of the open path are not closed the geometric phase is not manifestly gauge-invariant.

This paper aims at giving a new prescription for obtaining the geometric phase for non-cyclic, non-unitary and arbitrary quantum evolutions without explicitly closing the initial and final points by a geodesic. Using a 'reference section' of the bundle covering the curve in the projective Hilbert space we define the gauge-invariant and reparametrization-invariant geometric phase. To illustrate the result obtained, we calculate the non-cyclic geometric phase for a spin-$\frac{1}{2}$ particle undergoing arbitrary precession. Also, we consider a non-cyclic, non-unitary and non-Schrödinger evolution, such as a sequence of filtering measurements, to show the existence of the geometric phase.

Let $\mathscr{H}$ be the Hilbert space of dimension $n+1$, i.e. $\mathscr{H}=\mathbb{C}^{n+1}$ and the set of vectors $\{\psi\}\in\mathscr{H}$. Let $\mathscr{L}$ be the unit normed Hilbert space and the set of vectors $\{\psi/\|\psi\|\}\in\mathscr{L}$. The state of a quantum system is determined by a ray of the Hilbert space $\mathscr{H}$. The set of rays of $\mathscr{H}$ is called the projective Hilbert space $\mathscr{P}$. The projection map $\Pi:\mathscr{L}\to\mathscr{P}$ is a principal fibre bundle $\mathscr{L}(\mathscr{P}, U(1), \Pi)$, with structure group $U(1)$. This can be seen by considering the action of the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers on the space $\mathbb{C}^{n+1}-\{0\}$ given by the equivalence relation $(z_1, z_2, \ldots, z_{n+1})\lambda := (z_1\lambda, z_2\lambda, \ldots, z_{n+1}\lambda)\ \forall\ \lambda\in\mathbb{C}^*$. This is a free action and the orbit space is the space $\mathbb{C}P^n$ of complex lines in the Hilbert space $\mathscr{H}=\mathbb{C}^{n+1}$. Thus, we get the principal bundle $\mathbb{C}^*\to\mathbb{C}^{n+1}-\{0\}\to\mathbb{C}P^n=\mathscr{P}$ in which the projection map associates with each $n+1$ tuple $(z_1, z_2, \ldots, z_{n+1})$ the point in $\mathbb{C}P^n$ with the homogeneous coordinates

$(z_1, z_2, \ldots, z_{n+1})$. Thus, a quantum state at a given instant of time is represented by a point in $\mathscr{P}$ and the evolution of the system is given by a curve Γ in $\mathscr{H}$, which projects to a curve $\hat{\Gamma} = \Pi(\Gamma)$ in $\mathscr{P}$.

During a non-cyclic evolution the final state and the initial state do not differ merely by a phase factor because they belong to two different rays. The evolution curve is an open curve which lies in $\mathscr{H}$. The projection of the open curve $\Gamma: t \to |\psi(t)\rangle$ is $\Pi(\Gamma) = \hat{\Gamma}$ and it lies in $\mathscr{P}$. In the case of a cyclic evolution ($|\psi_f\rangle = \exp(i\phi)|\psi_i\rangle$) the closed curve $\hat{C}$ when lifted to $\mathscr{H}$, then the phase ϕ characterizes the gauge transformation necessary for closing C in $\mathscr{H}$. In the case of non-cyclic evolution the open path $\hat{\Gamma}$ if lifted to $\mathscr{H}$, then the initial and the final points correspond to two different rays and there is no such global gauge transformation to close these points. However, there is a way to compare the phases of the state vectors belonging to two different rays via the Pancharatnam [10] connection. It is simple, yet important Pancharatnam connection. The relative phase difference between the states $|\psi_i\rangle = |\psi(t_i)\rangle$ and $|\psi_f\rangle = |\psi(t_f)\rangle$ (if they are non-orthogonal) is given by

$$\exp(i[\Phi]_{t_i}^{t_f}) = \frac{\langle\psi_i|\psi_f\rangle}{|\langle\psi_i|\psi_f\rangle|}. \tag{1}$$

If $\langle\psi_i|\psi_f\rangle$ is real and positive, then the quantum system does not acquire any phase during an evolution from time t_i to t_f. This is the well known Pancharatnam connection. Here we emphasize that the phase difference given by (1) is true in general irrespective of closing the initial and the final points and it is the total phase acquired by a quantum system during an arbitrary quantum evolution between $[t_i, t_f]$, i.e.

$$\Phi_T = [\Phi]_{t_i}^{t_f} = \arg\left\langle \frac{\psi_i}{\|\psi_i\|} \middle| \frac{\psi_f}{\|\psi_f\|} \right\rangle. \tag{2}$$

We intend to give a prescription of how to separate out the geometric phase from the total phase $[\phi]_{t_i}^{t_f}$ which will be independent of the detail dynamics of the system and bring out its full geometric nature by showing its dependence uniquely only on the image of the curve $\hat{\Gamma}$ in the projective Hilbert space $\mathscr{P}$.

For the evolution under consideration, namely the non-cyclic evolution, let us consider the lift of an open path in $\mathscr{P}$ to $\mathscr{L}$, then there may be many open curves in $\mathscr{L}$. But there exists one special curve (which we have recognized for the first time) which is traced out by a 'reference section' in the $U(1)$ bundle of normed Hilbert space vectors over the projective Hilbert space $\mathscr{P}$. This section can be defined with respect to the initial state vector $|\psi(t_i)\rangle$ and the information about the geometric phase is obtained by integrating the inner product of the section and its path derivative from the initial point to the final point of the evolution curve.

To define this curve we consider a reference 'section' $|\chi_{t_i}(t)\rangle$ of the bundle covering $\rho(t) = \Pi(|\psi(t)\rangle)$. This is a map $s: \mathscr{P} \to \mathscr{L}$ such that the image of each point $\rho(t) \in \mathscr{P}$ lies in the fibre $\Pi(\rho)$ over ρ, i.e. $\Pi \circ s = \mathrm{id}_{\mathscr{P}}$, with the following properties:

(i) $\quad s\Pi(|\psi(t_i)\rangle) = |\psi(t_i)\rangle = |\chi_{t_i}(t_i)\rangle$

meaning sending $\Pi(|\psi(t_i)\rangle) \in \mathscr{P}$ to $|\psi(t_i)\rangle$ and $|\chi_{t_i}(t_i)\rangle$ and $|\psi(t_i)\rangle$ begin at the same point in the same ray.

(ii) $\quad \Pi(|\chi_{t_i}(t)\rangle) = \Pi(|\psi(t)\rangle) \qquad \forall\, t \in [t_i, t_f]$

i.e. the curves $\Gamma_i(t)$ and $\Gamma(t)$ project to the same open curve $\hat{\Gamma}$,

(iii) $\langle\chi_{t_i}(t_i)|\chi_{t_i}(t)\rangle=$ real and positive $\quad\forall\, t\in[t_i, t_f]$

i.e. at any later time t, $|\chi_{t_i}(t)\rangle$ keeps its phase unchanged with respect to the initial state $|\chi_{t_i}(t_i)\rangle$.

Keeping in mind the above properties we can see that the new 'section' (with respect to the initial point) is a mapping of the state curve $\hat{\Gamma}$ through the section s and is given by

$$|\chi_i(t)\rangle=|\chi_{t_i}(t)\rangle=\frac{\langle\psi(t)|\psi(t_i)\rangle}{|\langle\psi(t)|\psi(t_i)\rangle|}\frac{|\psi(t)\rangle}{\|\psi(t)\|}. \tag{3}$$

This 'reference section' of the bundle covering the curve $\hat{\Gamma}$ in $\mathscr{P}$ will be used to define the geometric phase for any arbitrary quantum evolution. In this sense equation (3) is an important step in our paper. Now it is easy to see that the following equation is obtained on differentiating (3):

$$\mathrm{i}\langle\chi_i(t)|\frac{\mathrm{d}}{\mathrm{d}t}|\chi_i(t)\rangle=\mathrm{i}\left(\frac{\langle\psi_i|\psi\rangle}{|\langle\psi_i|\psi\rangle|}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\langle\psi|\psi_i\rangle}{|\langle\psi|\psi_i\rangle|}\right)+\mathrm{i}\left\langle\frac{\psi}{\|\psi\|}\middle|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\psi}{\|\psi\|}\right)\right\rangle.$$

On integrating both sides from time t_i to t_f we have

$$\begin{aligned}\mathrm{i}\int_{t_i}^{t_f}&\left(\frac{\langle\psi_i|\psi\rangle}{|\langle\psi_i|\psi\rangle|}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\langle\psi|\psi_i\rangle}{|\langle\psi|\psi_i\rangle|}\right)\mathrm{d}t\\ &=-\mathrm{i}\int_{t_i}^{t_f}\left\langle\frac{\psi}{\|\psi\|}\middle|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\psi}{\|\psi\|}\right)\right\rangle\mathrm{d}t+\mathrm{i}\int_{t_i}^{t_f}\langle\chi_i|\frac{\mathrm{d}}{\mathrm{d}t}|\chi_i\rangle\,\mathrm{d}t.\end{aligned} \tag{4}$$

The dynamical phase for non-cyclic, not-unitary and non-Schrödinger evolutions of quantum systems is given by

$$[\Phi_d]_{t_i}^{t_f}=-\mathrm{i}\int_{t_i}^{t_f}\left\langle\frac{\psi(t)}{\|\psi(t)\|}\middle|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\psi(t)}{\|\psi(t)\|}\right)\right\rangle\mathrm{d}t. \tag{5}$$

Note that (5) reduces to the well known expression of the dynamical phase, namely $\delta=[\phi_d]_0^T=-1/\hbar\int_0^T\langle\psi(t)|H(t)|\psi(t)\rangle\,\mathrm{d}t$, for cyclic, unitary and Schrödinger evolutions of quantum systems.

The total phase (this is an important recognition in our derivation) is given by

$$\Phi_T=[\Phi]_{t_i}^{t_f}=\mathrm{i}\int_{t_i}^{t_f}\left(\frac{\langle\psi_i|\psi(t)\rangle}{|\langle\psi_i|\psi(t)\rangle|}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\langle\psi(t)|\psi_i\rangle}{|\langle\psi(t)|\psi_i\rangle|}\right)\mathrm{d}t=\arg\left\langle\frac{\psi_i}{\|\psi_i\|}\middle|\frac{\psi_f}{\|\psi_f\|}\right\rangle. \tag{6}$$

The desired expression for the geometric phase during an arbitrary quantum evolution is given by

$$[\Phi_g]_{t_i}^{t_f}=[\Phi]_{t_i}^{t_f}-[\Phi_d]_{t_i}^{t_f}$$

or

$$[\Phi_g]_{t_i}^{t_f}=\mathrm{i}\int_{t_i}^{t_f}\langle\chi_i(t)|\dot{\chi}_i(t)\rangle\,\mathrm{d}t=\mathrm{i}\int_{\mathrm{i}}^{\mathrm{f}}\langle\chi_i|\frac{\partial}{\partial\lambda^\mu}|\chi_i\rangle\,\mathrm{d}\lambda^\mu \tag{7}$$

where λ's are the coordinate in the projective Hilbert space $\mathscr{P}$.

That $[\Phi_g]_{t_i}^{t_f}$ is the geometric phase can be seen by virtue of its various properties. It is manifestly gauge-invariant. Because under $U(1)$ action $|\psi(t)\rangle \to e^{i\alpha(t)}|\psi(t)\rangle$, however, 'the reference section' $|\chi_i(t)\rangle$ undergoes a global gauge transformation by some fixed amount, i.e. $|\chi_i(t)\rangle \to e^{i\alpha(t_i)}|\chi_i(t)\rangle$ and hence the geometric phase remains the same. It is manifestly gauge-invariant not only under $U(1)$ action but also under a general transformation [11] of the type $|\psi(t)\rangle \to |\psi(t')\rangle = \zeta(t)|\psi(t)\rangle$, where $\zeta(t)$ is an arbitrary smooth complex function and $\zeta(t) \in \mathbb{C}^*$ with $|\zeta(t)| \neq 1$. Under this transformation $|\chi_i(t)\rangle$ transforms by a global phase factor and hence the geometric phase remains invariant. This is the beauty of the 'reference section' that it enables us to say that $[\phi_g]_{t_i}^{t_f}$ is a property of the projection of $|\psi(t)\rangle$ in $\mathscr{P}$ rather than of $|\psi(t)\rangle$ itself. It is also reparametrization-invariant, i.e. by changing the parameter from t to t' with $dt'/dt > 0$, the geometric phase remains unaltered and hence it is a property of only the unparametrized path $\hat{\Gamma}$ in $\mathscr{P}$. In addition to this, the real geometric phase is independent of the particular Hamiltonian used to evolve the quantum system along a given curve Γ in $\mathscr{S}$, rather it depends uniquely only on the curve $\hat{\Gamma}$ in $\mathscr{P}$. Unlike Samuel and Bhandari's [5] definition, our geometric phase has its own existence and physical meaning even if we do not close the end-points by a geodesic. It is important to note that the adiabatic Berry phase and the non-adiabatic AA phase can be considered as the special cases of the phase (7).

The above-mentioned properties, in fact, constitute the set of properties which characterize the geometric nature of some structures associated with an arbitrary quantum evolution. Thus, our expression (7) being an integral of a non-local integrand, provides for the first time a compact formula for the geometric phase during an arbitrary quantum evolution. Clearly, it is non-additive, in contrast to the dynamical phase which is a locally additive functional of Γ, being an integral along Γ of a locally defined integrand.

In defining the geometric phase for open curves we have used a (local) 'reference section'. It may give the impression that the geometric phase is of local nature. However, this is not so. We can see that the geometric phase (7) depends not only on the geometry of the path of evolution but also on the 'area' enclosed by a closed curve which the system has not visited (the closed curve is obtained by joining the end-points of the open curve by a shortest geodesic). In this sense the geometric phase is of non-local nature. Also, the presence of orthogonal normed vectors in the Hilbert space does not pose any problem. In order to validate (7) all we require is that the initial vector and the final vector should not be orthogonal because we have started from Pancharatnam's definition of phase which is valid only for non-orthogonal rays. For orthogonal rays the Pancharatnam phase becomes indeterminate (since no interference takes place between those rays). However, during the evolution, if the system passes through a point which is orthogonal to the initial point, then at that point the integrand is given by its limit along the curve and is finite [12]. Hence (7) is valid even if the system passes through an orthogonal point during its evolution.

It is sometimes said [13] that there are many ways of choosing the lift of $\hat{\Gamma}$, thereby making either total phase equal to zero (then $[\Phi_g]_{t_i}^{t_f} = -[\Phi_d]_{t_i}^{t_f}$) or dynamical phase equal to zero (then $[\Phi]_{t_i}^{t_f} = [\Phi_g]_{t_i}^{t_f}$). However, in this paper we have chosen a special lift of $\hat{\Gamma}$ in such a way that neither $[\Phi]_{t_i}^{t_f}$ nor $[\Phi_d]_{t_i}^{t_f}$ is zero and we still have a compact formula for the geometric phase. These above lines may give an impression that there are other choices of auxiliary connections which lead to different holonomies for a non-cyclic path. We can see clearly that this is not so. The local connection defined through a 'reference section' is the only one which is compatible with the non-cyclic, non-unitary and non-Schrödinger evolutions of quantum systems. This connection not only allows us to compare the geometric phase of two vectors in the same ray but also of two vectors (non-orthogonal) belonging to two different rays. For example, if we have a different lift (say) horizontal lift, then we will have the section $|\bar{\psi}(t)\rangle$ which undergoes parallel transportation. The section $|\bar{\psi}(t)\rangle$ is given by

$$|\bar{\psi}(t)\rangle = \exp\left(\mathrm{i}\int_{t_i}^{t} \mathrm{i}\left\langle \frac{\psi}{\|\psi\|}\middle|\frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{\psi}{\|\psi\|}\right)\right\rangle \mathrm{d}t'\right)\left|\frac{\psi(t)}{\|\psi(t)\|}\right\rangle. \tag{8}$$

Choosing this lift is equivalent to making the dynamical phase zero and the geometric phase is given by

$$\exp(\mathrm{i}[\Phi_g]_{t_i}^{t_f}) = \frac{\langle\bar{\psi}(t_i)|\bar{\psi}(t_f)\rangle}{|\langle\bar{\psi}(t_i)|\bar{\psi}(t_f)\rangle|}$$

or

$$[\Phi_g]_{t_i}^{t_f} = \arg\langle\bar{\psi}(t_i)|\bar{\psi}(t_f)\rangle. \tag{9}$$

But from our expression (3) (which is a different lift) we can see that the 'reference section' $|\chi_i(t)\rangle$ and the section $|\bar{\psi}(t)\rangle$ are related by

$$|\chi_i(t)\rangle = \frac{\langle\bar{\psi}(t)|\bar{\psi}(t_i)\rangle}{|\langle\bar{\psi}(t)|\bar{\psi}(t_i)\rangle|}\,|\bar{\psi}(t)\rangle. \tag{10}$$

Using the expression for the geometric phase (7) we get

$$\begin{aligned}[\Phi_g]_{t_i}^{t_f} &= \mathrm{i}\int_{t_i}^{t_f}\left(\frac{\langle\bar{\psi}(t_i)|\bar{\psi}(t)\rangle}{|\langle\bar{\psi}(t_i)|\bar{\psi}(t)\rangle|}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\langle\bar{\psi}(t)|\bar{\psi}(t_i)\rangle}{|\langle\bar{\psi}(t)|\bar{\psi}(t_i)\rangle|}\right)\mathrm{d}t \\ &= \arg\langle\bar{\psi}(t_i)|\bar{\psi}(t_f)\rangle. \end{aligned} \tag{11}$$

Here we have used the fact that $\langle\bar{\psi}(t)|\dot{\bar{\psi}}(t)\rangle = 0$. Thus, choosing different lifts, other expressions for the geometric phase can be found but all of them are identical to the one given by (7). Hence, the geometric phase defined in (7) for open paths is a unique one.

As an application of the above results we consider the example of a spin-$\frac{1}{2}$ particle undergoing arbitrary precession in a magnetic field. This is a simple yet non-trivial example. The state vector lives in a two-dimensional Hilbert space $\mathscr{H} = \mathbb{C}^2$ and the projective Hilbert space $\mathscr{P} = P_1(\mathbb{C})$ is the real two-dimensional sphere S^2. We can imagine a non-cyclic and non-unitary evolution of a spin state. The state vector at any time t is represented by a two-component spinor in (θ, ϕ) representation as

$$|\psi(t)\rangle = \begin{pmatrix}\cos\theta(t)/2\,\mathrm{e}^{\mathrm{i}\phi(t)/2}\\ \sin\theta(t)/2\,\mathrm{e}^{\mathrm{i}\phi(t)/2}\end{pmatrix}\mathrm{e}^{-\lambda\phi(t)} \tag{12}$$

where λ is a positive number. This is the simplest representation of a non-unitary evolution where the norm of the vector is not a constant; rather it changes with time as $\mathrm{d}\|\psi(t)\|/\mathrm{d}t = -\lambda\dot{\phi}\|\psi(t)\|$.

The 'reference section' $|\chi_i(t)\rangle$ is given by

$$|\chi_i(t)\rangle = \begin{pmatrix}\cos\theta(t)/2\,\mathrm{e}^{\mathrm{i}\alpha(t,t_i)}\\ \sin\theta(t)/2\,\mathrm{e}^{-\mathrm{i}\beta(t,t_i)}\end{pmatrix} \tag{13}$$

where

$$\alpha(t, t_i) = \phi(t)/2 - \tan^{-1}\left[\frac{\cos(\theta(t)+\theta_i)/2}{\cos(\theta(t)-\theta_i)/2}\tan(\phi(t)-\phi_i)/2\right]$$

and

$$\beta(t, t_i) = \phi(t)/2 + \tan^{-1}\left[\frac{\cos(\theta(t)+\theta_i)/2}{\cos(\theta(t)-\theta_i)/2}\tan(\phi(t)-\phi_i)/2\right].$$

For a non-cyclic and non-unitary evolution (θ goes from θ_i to θ_f and ϕ goes from ϕ_i to ϕ_f), the geometric phase calculated on using (7) is given by

$$[\Phi_g]_{t_i}^{t_f} = \tan^{-1}\left[\frac{\cos(\theta_f+\theta_i)/2}{\cos(\theta_f-\theta_i)/2}\tan(\phi_f-\phi_i)/2\right] - \frac{1}{2}\int_{t_i}^{t_f}\cos\theta\,d\phi. \tag{14}$$

The geometric phase will be equal to half the solid angle subtended at the centre of a unit sphere by a closed circuit obtained by joining the end-points of the open curve $\hat{\Gamma}$ by the shortest geodesic. At this point we can easily be convinced that for a cyclic quantum evolution the above geometric phase reduces to $[\phi_g]_0^T = \beta(C) = \frac{1}{2}\oint(1-\cos\theta)\,d\phi$, which is nothing but the Aharonov and Anandan phase for a spin-$\frac{1}{2}$ particle.

To better illustrate the result, we consider the example of a non-unitary evolution, such as a sequence of filtering measurements. A beam of particles polarized along z with initial vector $|z\rangle$ is split into two orthogonal components ($|x\rangle$ and $|\bar{x}\rangle$), i.e. $|x\rangle\langle x|z\rangle + |\bar{x}\rangle\langle\bar{x}|z\rangle$. When this passes through a filter we have filtered component $|x\rangle\langle x|z\rangle$ (the component $|\bar{x}\rangle\langle\bar{x}|z\rangle$ is discarded and the intensity is reduced by $\frac{1}{2}$). Suppose $|x\rangle\langle x|z\rangle$ passes through another apparatus oriented along y (in which the $|\bar{y}\rangle$ component is discarded and the intensity is reduced by $\frac{1}{4}$). The filtered component is $|y\rangle\langle y|x\rangle\langle x|z\rangle$. We do not send this component through another apparatus oriented along z, so we have a non-cyclic evolution. Thus we can represent the sequence of filtering measurements as

$$|z\rangle \to |x\rangle\langle x|z\rangle \to |y\rangle\langle y|x\rangle\langle x|z\rangle$$

or

$$|\psi_1\rangle \to |\psi_2\rangle \to |\psi_3\rangle.$$

During this non-cyclic, non-unitary and non-Schrödinger evolution the initial and final states differ by a phase

$$\Phi = \arg\langle z|y\rangle\langle y|x\rangle\langle x|z\rangle = -\arg\langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_1\rangle = \arg\Delta^{(3)} \tag{15}$$

where $\Delta^{(3)} = \langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_1\rangle$ is a complex 3-point Bargmann invariant [13]. We will show that this phase Φ is purely of geometric nature. From our definition of the geometric phase (7) we can show [14] that the phase acquired by the system during an evolution from point $\Pi(\psi_1)$ to point $\Pi(\psi_2)$ plus the phase acquired from point $\Pi(\psi_2)$ to $\Pi(\psi_3)$ is not the same as that of the phase acquired by the system during an evolution from point $\Pi(\psi_1)$ to $\Pi(\psi_3)$. The excess geometric phase (this shows the non-integrability nature of the phase) is given by

$$[\Phi_g]_1^3 - \{[\Phi_g]_1^2 + [\Phi_g]_2^3\} = -\arg\langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_1\rangle. \tag{16}$$

In the above filtering experiment, since $|\psi_1\rangle$ is in phase with $|\psi_2\rangle$ and $|\psi_2\rangle$ is in phase with $|\psi_3\rangle$ we have $[\phi_g]_1^2=0$ and $[\phi_g]_2^3=0$. This shows that the phase difference Φ in the filtering experiment is nothing but the geometric phase acquired by the system in going from a point $\Pi(\psi_1)$ to a point $\Pi(\psi_3)$, i.e. $\Phi=[\Phi_g]_1^3$.

To make the general geometric phase more useful we outline (without going into much detail) how to measure this non-cyclic phase in an interference experiment. In an interferometric setup, the incident wave (it could be, say, a neutron beam) $|\psi_i\rangle$ is split coherently with a beam splitter into two equal subwaves. A refractive index material in the path of subwave 1 can be kept to cause a $U(1)$ phase change β of the state. Subwave 2 acts as a reference beam. The two subwaves $|\psi_1\rangle=\exp(i\beta)|\psi_i\rangle$ and $|\psi_2\rangle=|\psi_i\rangle$ recombine to produce an interference pattern.

$$I(\beta)\propto\tfrac{1}{2}\||\psi_1\rangle+|\psi_2\rangle\|^2\propto(1+\cos\beta). \tag{17}$$

Now a magnetic field is introduced in path 1 of the interferometer to affect a non-cyclic parallel transportation [15] of the state $|\psi_1\rangle$ to $R(\phi)|\psi\rangle$. Here, $R(\phi)$ is a rotation operator which does just this job. When we recombine the parallel transported state with subwave 2 an interference pattern is produced

$$I(\beta,\Phi)\propto\tfrac{1}{2}\|R(\phi)|\psi_1\rangle+|\psi_2\rangle\|^2\propto(1+\mathrm{Re}\langle\psi_2|R(\phi)|\psi_1\rangle)$$

or

$$I(\beta,\Phi)\propto(1+|\langle\psi_i|\psi_f\rangle|\cos(\beta+\Phi)). \tag{18}$$

Here, $\phi=\Phi_g$ is the non-cyclic geometric phase acquired by subwave 1 while undergoing parallel transportation and $\langle\psi_i|R(\phi)|\psi_i\rangle=\langle\psi_i|\psi_f\rangle=|\langle\psi_i|\psi_f\rangle|\exp(i\Phi_g)$. Thus, comparing the two interference patterns we can infer the geometric phase Φ_g. The shift $-\phi$ of the interference pattern recorded with and without the phase object as a function of auxiliary phase β is crucial in determining the non-cyclic geometric phase. The measurement of non-cyclic phase via the interferometer method and polarimeter method is discussed by Wagh and Rakhecha [16].

To summarize our findings, we present a compact expression for the geometric phase in the case of non-cyclic, non-unitary and non-Schrödinger evolutions (in fact, for all arbitrary quantum evolutions) without explicitly closing the end-points by a geodesic. This non-local phase is inherently present in all quantum evolutions. Also, we have shown that the geometric phase defined in this paper is a unique one and it reduces to the known definition of the geometric phase in the case of cyclic, unitary and Schrödinger evolutions of quantum systems. Two non-trivial examples are studied to realize the geometric phase. Detail discussion and relation between the non-cyclic geometric phase, the quantum metric tensor [17–19] and the length of the curve [20] will be reported elsewhere [14].

Acknowledgments

I thank Dr S V Lawande and Dr S R Jain for useful discussions. My thanks go to Professor N Mukunda and Professor J S Anandan for illuminating some points in the paper. I also thank Dr A G Wagh for useful discussions concerning the experimental measurement of the non-cyclic phase.

References

[1] Berry M V 1984 *Proc. R. Soc.* A **392** 45
[2] Simon B 1983 *Phys. Rev. Lett.* **51** 2167
[3] Aharonov Y and Anandan J 1987 *Phys. Rev. Lett.* **58** 1593
[4] Anandan J 1991 *Found. Phys.* **21** 1256
[5] Samuel J and Bhandari R 1988 *Phys. Rev. Lett.* **60** 2339
[6] Shapere A and Wilczek F (ed) 1989 *Geometric Phases in Physics* (Singapore: World Scientific)
Vinitskii S I *et al* 1990 *Sov. Phys. Usp.* **33** 403
[7] Pati A K 1991 *Phys. Lett.* **159A** 105
[8] Pati A K 1992 *J. Phys. A: Math. Gen.* **25** L1001
[9] Pati A K and Joshi A 1993 *Phys. Rev.* A **47** 98
[10] Pancharatnam S 1956 *Proc. Indian Acad. Sci.* A **44** 247
[11] Aitchison I J R and Wanelik K 1992 *Proc. R. Soc.* A **439** 25
[12] Wagh A G and Rakhecha V C 1993 *Pramana—J. Phys.* **41** 246
[13] Mukunda N and Simon R 1993 *Ann. Phys.* **228** 205
[14] Pati A K 1994 *Phys. Rev.* A (submitted)
[15] Wagh A G and Rakhecha V C 1993 *Phys. Rev.* A **48** R1729
[16] Wagh A G and Rakhecha V C 1995 *Phys. Lett.* A (submitted)
[17] Provost J P and Valle G E 1981 *Commun. Math. Phys.* **76** 289
[18] Anandan J and Aharonov Y 1990 *Phys. Rev. Lett.* **65** 1697
[19] Anandan J 1990 *Phys. Lett.* **147A** 3
[20] Pati A K 1994 *Pramana—J. Phys.* **42** 455